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A mathematical theory of synchronization by self-feedback

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Abstract. In this paper, we develop a theoretical analysis of the synchronization of two coupled systems by self-feedback. By means of Lyapunov direct method, we reveal some mechanisms for synchronization and present some criteria for synchronizability, which can be used to deal with a wide range of systems.

1. Introduction

The synchronization of two nonlinear dynamical systems is a phenomenon of importance in a wide range of applied sciences, and has received much attention and investigation in the past decades [1–7]. In particular, it has been shown recently that two identical systems, or homosystems, can be synchronized by some form of coupling. One common procedure is to introduce a self-feedback of the form $k(x-y)$, i.e. the difference between the current states of the two systems is used as an inhibitory effect on the separation of orbits. Much work, mainly numerical, has been done along these lines [8–12].

The purpose of this paper is to develop a mathematical theory for this phenomenon with the hope that one would be able to gain greater understanding of the mechanimisms underlying synchronization and to cope with a broader range of systems.

2. Preliminaries

Consider two systems described by ordinary differential equations

$$\dot{x} = f(x,\alpha) \qquad x = (x_1, \cdots x_n) \in R^n \tag{1}$$

$$\dot{y} = g(y,\beta) \qquad y = (y_1, \cdots y_n) \in R^n \qquad f, g \in C^1(R^n, R^n) \tag{2}$$

where α and β are parameters which can be controlled in practice. The coupling of the two systems is described by

$$\begin{aligned} \dot{x} &= f(x,\alpha) + \eta(x-y) \\ \dot{y} &= g(y,\beta) + \xi(x-y) \end{aligned} \tag{3}$$

where η and ξ are called coupling functions. In practice, η and ξ usually have the form $k(y-x)$ and $\bar{k}(x-y)$, respectively, where k and $\bar{k}$ are diagonal matrices with positive

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elements. The $\max k_i (1 \leqslant i \leqslant n)$ and $\max \bar{k}_i (1 \leqslant i \leqslant n)$ are called the coupling strengths of k and $\bar{k}$, respectively.

Definition 2.1. Let $(x(t, x_0), y(t, y_0))$ be a solution to the system (3) with initial values (x_0, y_0). If $x(t, x_0) = y(t, y_0)$, then $(x(t, x_0), y(t, y_0))$ is called a synchronized solution. If $|x(t, x_0) - y(t, y_0)| \to 0$ as $t \to \infty$, then equation (3) is said to be synchronized through the coupling $\eta(x - y)$ and $\xi(x - y)$.

Since this kind of ideal synchronization cannot be fully realized in practice, a more practical definition of synchronization should be that defined as follows.

Definition 2.2. Given an $\varepsilon > 0$, if $\|x(t, x_0) - y(t, y_0)\| < \varepsilon$, then $(x(t, x_0), y(t, y_0))$ is called an ε-synchronized solution. (or noisy synchronization as called in [12]). If there exists $T > 0$, such that $|x(t, x_0) - y(t, y_0)| \leqslant \varepsilon$ for $t > T$, then the system (3) is said to be ε-synchronizable by $\eta(x, y)$ and $\xi(x, y)$.

In the case where $g(y, \beta) = f(y, \alpha)$, the system (3) is referred to as two identical coupled systems. In the case where g is the same as f with $\beta \neq \alpha$, the system (3) is referred to as the coupling of two homo-systems [12]. Otherwise the system (3) is called two coupled hetero-systems.

3. Main results

For convenience we omit the parameters α, β in the following discussion.

By making the change of variables

$$\bar{x} = x - y \qquad \bar{y} = x + y \qquad x, y \in R^n$$

we can transform (3) into

$$\begin{aligned} \dot{\bar{x}} &= f\left(\frac{\bar{x} + \bar{y}}{2}\right) - g\left(\frac{\bar{y} - \bar{x}}{2}\right) + \eta(\bar{x}) - \xi(\bar{x}) \\ \dot{\bar{y}} &= f\left(\frac{\bar{x} + \bar{y}}{2}\right) + g\left(\frac{\bar{y} - \bar{x}}{2}\right) + \eta(\bar{x}) + \xi(\bar{x}). \end{aligned} \tag{4}$$

In view of definition 2.1, if the system (3) possesses a synchronized solution $x(t, x_0) = y(t, y_0)$ for every initial value $x_0 = y_0$, then the hyperplane $x - y = 0$ is required to be a stable invariant manifold of (3), or equivalently the hyperplane $\bar{x} = 0$ is required to be the invariant manifold of the system (4).

From this observation, we have the first assertion from the first expression of (4).

Proposition 3.1. The hyperplane $x - y = 0$ is an invariant manifold of (3), if and only if f and g are identical.

From proposition 3.1, we see that identity of two systems is the necessary condition for sychronization.

Proposition 3.2. Suppose that f and g are identical, and there exists a constant $c > 0$ such that

$$(x - y) \cdot \eta(x - y) - (x - y) \cdot \xi(x - y) \leqslant -c\|x - y\|^2$$

and

$$\nu^{\mathrm{T}} J f(x) \nu \leqslant \bar{c}\|\nu\|^2 \qquad \forall\ \nu \in R^n \quad x \in R^n \quad \bar{c} < c$$

then the hyperplane $x - y = 0$ is asymptotically stable, i.e. every orbit near $x - y = 0$ approaches $x - y = 0$ as $t \to \infty$.

To see this, consider the derivative of the function $V(x) = \frac{1}{2}\|x\|^2$ along the orbits of (4):

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \overline{x}\cdot\dot{\overline{x}} = \overline{x}\cdot\left[f\left(\frac{\overline{x}+\overline{y}}{2}\right) - g\left(\frac{\overline{y}-\overline{x}}{2}\right)\right] + \overline{x}\cdot(\eta(\overline{x}) - \xi(\overline{x}))$$

$$= \overline{x}^{\mathrm{T}} Jf\left(\frac{\overline{y}}{2}\right)\overline{x} - c\|\overline{x}\|^2 + \mathrm{O}(\|\overline{x}\|^3). \quad (5)$$

In the light of the stability theory of dynamical systems, (5) guarantees that every orbit with an initial value sufficiently near $\overline{x} = 0$ will approach it, which is what we hope for.

Denote $\frac{1}{2}(Jf(\overline{y}/2) + Jf(\overline{y})^{\mathrm{T}})$ by $A(\overline{y})$. Clearly from (5) we have the following result.

Proposition 3.3. Let $\lambda_{\max}(\overline{y})$ be the maximal eigenvalue of $A(\overline{y})$. If $\lambda_{\max} < c$, then $\overline{x} = 0$ is asymptotically stable and consequently $x - y = 0$ is asymptotically stable.

Remarks. From proposition 3.2 we can see that if we want a coupling like $k(x - y)$ to be very weak, i.e. k can be very small, while the coupling can take effect, then we have to impose some conditions on the original systems, say, $f(x)$ here. One of them is that the Lyapunov exponent with respect to $f(x)$ is not greater than 0. Actually this is rare in practical physical systems. However, if there exists an attractive basin B (a region B such that the nearby orbits will enter it as $t \to \infty$) then some conditions can be imposed on $f(x)$ only in the region B to guarantee the synchronization in B; this is usually the case in the Lorenz system [12].

Let Ω be a bounded domain of R^n; to relax the idential condition on $f(x)$ and $g(x)$, we only assume that $f(x) = g(x), \forall\, x \in \Omega$ and call them Ω-identical systems. To establish a criterion for synchronization in this case we first define a subset in $R^n \times R^n$ as follows.

Let $\operatorname{diag}\Omega = \{(x, y) \in R^n \times R^n | x = y, y \in \Omega\}$, the cylinder C_δ being defined as $C_\delta = \{p + t\boldsymbol{n} |\ \ p \in \operatorname{diag}\Omega,\ \ 0 \leqslant t < \delta\}$, where $\boldsymbol{n}$ is a unit vector perpendicular to $\operatorname{diag}\Omega$.

Now we have the following proposition.

Proposition 3.4. Suppose that (i) the coupling functions ξ and η satisfy

$$(x - y)\cdot\eta(x - y) - (x - y)\cdot\xi(x - y) \leqslant -c\|x - y\|^2 \qquad c > 0.$$

(ii) f (and g) satisfies

$$\nu^{\mathrm{T}} Jf(x)\nu < c\|\nu\|^2 \qquad x \in \Omega \quad \nu \in R^n.$$

(iii) $f(x)\cdot\boldsymbol{e} > 0, \forall\, x \in \partial\Omega$, where $\partial\Omega$ is the boundary of Ω in R^n and $\boldsymbol{e}$ is the unit normal vector field on $\partial\Omega$ which points inward. Then (1) and (2) can be synchronized by η and ξ in $\operatorname{diag}\Omega$.

This proposition can be roughly proved as follows. On the one hand, from conditions (i)–(iii) it follows that there exists a $\delta > 0$ such that the cylinder C_δ defined above is an attractive basin: every orbit with initial point sufficiently near C_δ will enter C_δ and remain in it. On the other hand, these conditions guarantee that no set in C_δ other than that contained in $\operatorname{diag}\Omega$ can be an invariant set under the flow of (3). Now the invariance principle in the stability theory of dynamical systems [13, 14] ensures that synchronization occurs somewhere in $\operatorname{diag}\Omega$.

Now let us see what mechanism underlies the so-called ε-synchronization. For the function $V(\overline{x}) = \frac{1}{2}\|\overline{x}\|^2$, its derivative along the orbits of (3) is

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \overline{x}\cdot f\left(\frac{\overline{x}+\overline{y}}{2}\right) - \overline{x}\cdot g\left(\frac{\overline{y}-\overline{x}}{2}\right) + \overline{x}\cdot\eta(\overline{x}) - \overline{x}\cdot\xi(\overline{x})$$

$$= \bar{x}\cdot\left(f\left(\frac{\bar{y}}{2}\right)-g\left(\frac{\bar{y}}{2}\right)\right)+\frac{1}{2}\bar{x}^{\mathrm{T}}\left(Jf\left(\frac{\bar{y}}{2}\right)+Jg\left(\frac{\bar{y}}{2}\right)\right)\bar{x}$$

$$+\bar{x}\cdot\eta(\bar{x})-x\cdot\xi(\bar{x})+\mathrm{O}(\|\bar{x}\|^3). \tag{6}$$

Let $A_{fg} = \left(Jf(\bar{y}/2)+Jg(\bar{y}/2)\right)^{\mathrm{T}}+\left(Jf(\bar{y}/2))+Jg(\bar{y}/2)\right)$, and $\lambda_{\max}$ = the maximal eigenvalue of A_{fg}, suppose $\bar{x}\cdot\eta(\bar{x})-\bar{x}\xi(\bar{x})\leqslant -c\|\bar{x}\|^2$. If $\lambda_{\max}<2c$, then

$$\frac{\mathrm{d}V}{\mathrm{d}t}\leqslant\|\bar{x}\|\left\|f\left(\frac{\bar{y}}{2}\right)-g\left(\frac{\bar{y}}{2}\right)\right\|-\left(c-\frac{1}{2}\lambda_{\max}\right)\|\bar{x}\|^2+\mathrm{O}(\|x\|^3)$$

so, if we want (1) and (2) to be ε-synchronized, then $f(x)-g(x)$ must satisfy $\|f(x)-g(x)\|<(c-\frac{1}{2}\lambda_{\max})\varepsilon$ somewhere in the synchronized region.

All of this can be put in the following proposition.

Proposition 3.5. A necessary condition that (1) and (2) can be ε-synchronized in a region $\Omega\subset R^n$ by the coupling functions η and ξ is that

$$|f(x)-g(x)|<b(\lambda_{\max},c)\varepsilon$$

where b is a number dependent on $\lambda_{\max}$ and the coupling strength c.

Therefore if we want two systems to be ε-synchronizable in a region, then we must be sure that the difference between the two systems is 'as small as ε' in this region.

Mathematically, we can have a more general theorem which can easily be proved by means of stability theory.

Theorem. Let $V(\bar{x}) : R^m\to R$ be a positive definite function, i.e. $V(0)=0$, $V(\bar{x})>0$, $\bar{x}\neq 0$. Given two desired coupling function $\bar{\eta}(\bar{x})$ and $\xi(\bar{x})$, if there exists such a positive definite function $V(\bar{x})$ such that the derivative of $V(\bar{x})$ along (4) satisfies

(i)
$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4)}=\mathrm{grad}\,V\cdot\left[f\left(\frac{\bar{x}+\bar{y}}{2}\right)-g\left(\frac{\bar{y}-\bar{x}}{2}\right)\right]+\mathrm{grad}\,V\cdot[\bar{\eta}(\bar{x})-\xi(\bar{x})]$$
$$\leqslant -U(\bar{x})$$

where $\mathrm{grad}\,V=(\partial V/\partial\bar{x}_1,\ldots,\partial V/\partial\bar{x}_n)$, and $U(\bar{x}) : R^n\to R$ is also a positive definite function. Then (1) and (2) can be synchronized by $\eta(x-y)$ and $\xi(x-y)$ in the case of f and g are identical;

(ii)
$$\left.\frac{\mathrm{d}V}{\mathrm{d}t}\right|_{(4)}<0 \qquad \forall\,|\bar{x}|>\varepsilon$$

then (1) and (2) can be ε-synchronized by $\eta(x-y)$ and $\xi(x-y)$.

Remarks. What we have discussed up to now is just local (ε)-synchronization, i.e. the asymptotic properties of the solutions of (3) in the vicinity of the plane $x=y$. We hope to know what can be said about global (ε)-synchronization, and this is another theme to be discussed later.

4. Applications to chaotic system

In this section we utilize what was developed in section 3 to dicuss synchronization in chaotic systems instead of the usual numerical methods. To save tedious calculation, we just give an outline of the discussion.

Consider the Lorenz system

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx - y - xz\\ \dot{z} &= xy - bz\end{aligned} \tag{7}$$

and its copy system with feedback

$$\begin{aligned}\dot{\overline{x}} &= \sigma(\overline{y}-\overline{x}) + c(x-\overline{x})\\ \dot{\overline{y}} &= r\overline{x} - \overline{y} - \overline{x}\,\overline{z} + c(y-\overline{y})\\ \dot{\overline{z}} &= \overline{x}\,\overline{y} - b\overline{z} + c(z-\overline{z}).\end{aligned} \tag{8}$$

Let us see for what c the synchronization occurs between (8) and (7).

As shown by Sparrow [15], the ellipsoid E defined by

$$A = rx^2 + \sigma y^2 + \sigma(z-2r)^2 \leqslant m + \xi(\xi = \text{constant} > 0)$$

is an attractive bounded invariant set, where $m(r,\sigma)$ is the maximum value of A on the full ellipsoid bounded by

$$rx^2 + y^2 + b(z-2r)^2 = 4br^2.$$

Now for (7), we have

$$Jf(x) = \begin{pmatrix} -\sigma & \sigma & 0\\ r-z & -1 & -x\\ y & x & -b \end{pmatrix}. \tag{9}$$

The maximum eigenvalue of $(Jf(x) + (Jf(x)^{\mathrm{T}})/2$, denoted by $\lambda_{\max}(\sigma, r, b, m+\xi)$, can be calculated. Now take $c > \lambda_{\max}(\sigma, r, b, m+\xi)$ in (8). It follows from proposition 3.3 and its ensuing remarks (or proposition 3.4) that (8) can synchronized with (7) in E for c chosen above.

The ϵ-synchronization is more practical; however, little theoretical work has been done on this problem as far as we know. Here we touch on it.

Consider a homosystem of (7) with feedback

$$\begin{aligned}\dot{\overline{x}} &= \overline{\sigma}(\overline{y}-\overline{x}) + c(x-\overline{x})\\ \dot{\overline{y}} &= \overline{r}\,\overline{x} - \overline{y} - \overline{x}\,\overline{z} + c(y-\overline{y})\\ \dot{\overline{z}} &= \overline{x}\,\overline{y} - \overline{b}\,\overline{z} + c(z-\overline{z}).\end{aligned} \tag{10}$$

Then in terms of (3), we have

$$\|f-g\| \leqslant \left((\overline{\sigma}-\sigma)^2 + (\overline{b}-b)^2 + (\overline{r}-r)^2\right)^{1/2} d$$

on $E \cup \overline{E}$, where E and $\overline{E}$ are ellipsoids corresponding to two groups of parameters $\sigma, r, b, \overline{\sigma}$, $\overline{r}$ and $\overline{b}$, respectively, and d is the diameter of $E \cup \overline{E}$.

Keeping in mind that the smaller the difference between the two groups of the parameters the smaller the diameter d, so we can fix d for given E and $\overline{E}$, then the homosystem (10) can hold uniformly in the case of the smaller difference between the two groups of parameters.

Denote by $\lambda_{\max}$ the maximal eigenvalue of A_{fg} over $E \cup \overline{E}$, where A_{fg} is defined as in (6), take a proper coupling constant $2c > \lambda_{\max}$. If

$$((\overline{\sigma}-\sigma)^2 + (\overline{b}-b)^2 + (\overline{r}-r)^2)^{1/2} < (c - \lambda_{\max}/2)\epsilon/d$$

then in the light of proposition 3.5 and the arguments preceding it, (9) can ϵ-synchronize with (7).

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